Pre-service Teachers' Approach Preferences in Geometry Problems: Analytic, Synthetic or Vectorial?

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ABSTRACT
The aim of this study is to examine the effect of the learning environment designed based on different approaches on the change in pre-service teachers' approach preferences and geometry achievement in geometry problems. Twenty pre-service mathematics teachers were involved in geometry lessons for 10 weeks in a learning environment designed based on Analytic Approach (AA), Synthetic Approach (SA) and Vectorial Approach (VA). There were six geometry problems in the pre-test and post-test administered before and after the implementation, respectively. The solutions of the problems in these tests were evaluated by means of a rubric. Then, two pre-service teachers with "high", "medium" and "low" achievement levels were identified based on the scores obtained from these tests and clinical interviews were conducted with them. As a result of the study, it was determined that the most preferred approach before and after the implementation was SA. However, it was determined that AA and VA, which were less preferred before the implementation, were preferred more after the implementation. In addition, the designed learning environment contributed positively to pre-service teachers' geometry achievement. It was determined that the success of the pre-service teachers in the post-test solutions increased considerably compared to the pre-test. It is thought that it is important to design other courses in the university, for example; Analytic Geometry course should be supported with SA and VA or the Algebra course should be supported with AA and VA and presented to prospective teachers in a new structure.
Keywords: Analytic approach, Synthetic approach, Vectorial approach, Learning environment, Geometry problems

INTRODUCTION
Geometry is accepted as an important field within the scope of mathematics education from primary school level to university level because it has an important implementation area in other sub-fields of basic mathematics, provides people with the power of spatial perception and develops problem solving skills by activating the mind (Baki, 2002; Sherard, 1981; Temur, 2007). Considering both the general aims of geometry education and the qualities that geometry education provides to the individual, the emphasis on the development and contribution of geometry to the development of problem solving skills in individuals draws attention. The National Council of Teachers of Mathematics (NCTM, 2000), which draws attention to the development of geometry in some skills of students, expresses geometry as a field that develops students' problem solving, reasoning, association, communication and verification skills. According to NCTM (2000), thanks to these skills created by geometry in individuals, students can activate their minds, analyze and solve problems, make comparisons and establish connections between mathematics and life. In addition, geometry offers an implementation area for the development of students' inference and proof skills (NCTM, 2000). According to Ball (1988), in the mathematics teaching process, the subfields of mathematics are presented to students in discrete sections and students are rarely encouraged to make connections between the different ideas they have learnt, and situations where students can solve problems by making connections between different approaches are not created in learning environments. However, one of the ways of constructing mathematical knowledge is connectedness. According to NCTM (2000), making use of the sub-fields of mathematics helps students to establish a stronger connection between their mathematical knowledge. Establishing mathematical connections between subfields (e.g., between different concepts, their different representations, different topics and different areas within mathematics, as well as between mathematics and other subjects) is a very important part of mathematical understanding (Dreyfus & Eisenberg, 1986; Hiebert & Carpenter, 1992; Kieren, 1990; Sfard, 1991; Sierpinska, 1994; Skemp, 1987). A person who cannot establish this connection has to memorize many different concepts and methods. Making connections between the mathematical ideas that an individual has means combining new ideas with related ideas and solving mathematical situations by using similar concepts and methods that will be useful in new situations.

When the literature is examined, it is seen that the studies on the solutions of geometry problems generally consist of using a single approach in solving problems (Barbeau, 1988; Dindyal, 2003; Kwon, 2012; Pambuccian, 1993). However, it can be thought that a learning environment that offers a broader perspective that can affect students' problem solving processes will be created by using problems that can be solved through different approaches within the scope of geometry course. In this way, the superior aspects of the solutions of a problem can be revealed more clearly if they are supported by the implementations made with the students.

Geometry problems in which different approaches are used in their solutions are accepted as an effective teaching tool that will enable students to establish the connection between geometry and other fields (House & Coxford, 1995; Leikin, 2003, 2007; NCTM, 2000; Polya, 1973; Schoenfeld; 1983; Silver, 1997). If we want to look deeper into the teaching activities that support solving problems using more than one approach, Leikin et al. (2006) and Leikin and Levav-Waynberg (2007) used problems that require finding the solution of a given geometry problem using more than one approach in their studies. The difference between the solutions of such problems is indicated as different representations of the mathematical concept, different properties of mathematical concepts (definitions, theorems, auxiliary elements, etc.) or the use of theorems and tools belonging to subfields of mathematics (Leikin, 2007; Leikin & Levav-Waynberg, 2007; Leikin & Levav-Waynberg, 2008). Analytical Approach (AA), Synthetic Approach (SA) and Vectorial Approach (VA), which use definitions, theorems, etc. in the subfields of geometry and algebra, respectively, can offer the opportunity to look at the solutions of geometry problems from a different perspective. With this different perspective gained in the solutions of geometry problems, students will be able to establish a connection between the mathematical knowledge they have and, in this process, they will have the opportunity to meet with learning environments where different approaches are used together in geometry problem solving.

AA is a method that allows to solve geometric problems by using the coordinate system, the discovery of Decartes. In this approach, geometric properties are combined with algebraic methods. Here, students are asked to choose the coordinate system that will make the solution as comprehensible as possible and to interpret the equations obtained in terms of the problem.

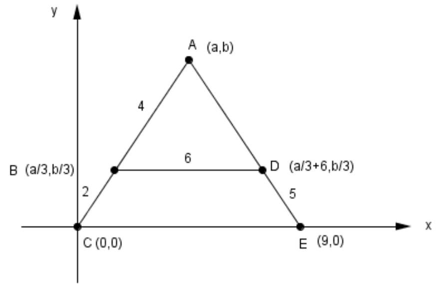
To solve problems in this approach;
1. Distance between two points,
2. Equation of a line given a point and slope,
3. Equation of a line with two given points,
4. Lines with equal slopes are parallel,
5. If two lines are perpendicular, the product of their slopes is -1
basic features such as the basic features should be known at least to a minimum extent (French, 2004).

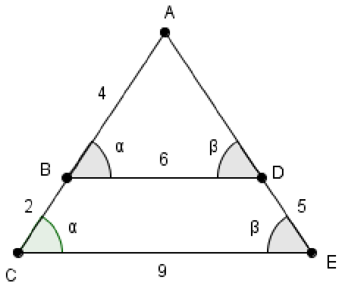
The teaching of Euclidean geometry has a very important place at every stage at school level. On the other hand, the problem solving approach using Euclidean geometry encountered at every stage of the education period is defined as SA. This method is based on axioms, postulates and theorems (Ministry of National Education [MoNE], 2010). In geometry problems solved through this approach, axioms, postulates and theorems of geometry are used. When the studies involving problem solving in geometry are analysed, it is seen that different paths belonging to only one approach are generally used and different paths belonging to SA or AA are used in geometry problem solving, but the connectivity between approaches is not mentioned (Levav-Waynberg, 2011; Levav-Waynberg & Leikin, 2006; Sierpinska, 2000; Silver et al., 2005). On the other hand, there are very few studies in which VY is used in geometry problem solving. However, including the connectedness feature, which has an important place in the construction of mathematical knowledge, in the implementations in the problem solving process and reconsidering the courses carried out in this context may contribute to the development of students in problem solving processes.

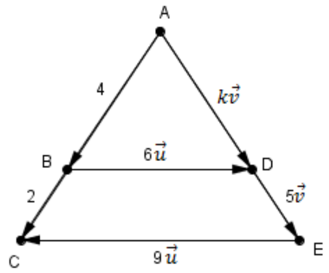
Vectorial Approach is mainly based on knowledge in algebra and vector space. This approach, which is based on combining geometry and algebra, has gradually taken its place in many curricula. Some researchers agree that plane geometry is too much dominated by Euclidean geometry. At the Cambridge Conference of School Mathematics (1963), a view was presented that “there are many ways of teaching geometry which can be followed, and which have their own advantages”. One of these is VA. While the coordinate plane is utilized in the representation of vectors, it is also seen that implementations through matrices are also used. In the following, it is discussed how a problem can be answered with these three approaches:

Problem: In the figure below, if BD // CE is |AB|=4,|BC|=2,|BD|=6,|DE|=5 and |CE|=9, then |AD|= ?

Solution:

<table>
<tr><th>Solution with Analytical Approach</th><th>Solution with Synthetic Approach</th><th>Solution with Vectorial Approach</th></tr>
<tr>
<td>


$$|DE| = \sqrt{\left(\frac{a}{3}+6-9\right)^2+\left(\frac{b}{3}-0\right)^2} = 5$$
$$= \left(\frac{a}{3}-3\right)^2+\left(\frac{b}{3}\right)^2 = 25 \quad \dots(1)$$
$$|BC| = \sqrt{\left(\frac{a}{3}-0\right)^2+\left(\frac{b}{3}-0\right)^2} = 2$$
$$= \left(\frac{a}{3}\right)^2+\left(\frac{b}{3}\right)^2 = 4 \quad \dots(2)$$
$$(1) \Longrightarrow \left(\frac{a}{3}\right)^2 - 2a + 9 + \left(\frac{b}{3}\right)^2 = 25$$
$$-2a + 13 = 25$$
$$a = -b \quad \dots(3)$$
$$|AB| = \sqrt{\left(a-\frac{a}{3}\right)^2+\left(b-\frac{b}{3}\right)^2} = 4$$
$$= \left(\frac{2a}{3}\right)^2+\left(\frac{2b}{3}\right)^2 = 16 \quad \dots(4)$$
$$|AD| = \sqrt{\left[a-\left(\frac{a}{3}+6\right)\right]^2+\left(b-\frac{b}{3}\right)^2}$$
$$= \sqrt{\left(\frac{2a}{3}-6\right)^2+\left(\frac{2b}{3}\right)^2}$$
$$= \sqrt{\left(\frac{2a}{3}\right)^2-8a+36+\left(\frac{2b}{3}\right)^2}$$
$$From\ (3)\ ve\ (4)$$
$$\sqrt{16+48+36} = 10\ cm$$
</td>
<td>


Since BD // CE
$$m(\widehat{ABD}) = m(\widehat{ACE})$$
$$m(\hat{A}) = m(\hat{A})$$
$$m(\widehat{ADB}) = m(\widehat{AEC})dir$$
Then from the Angle-Angle-Angle Similarity Theorem;
$$\overset{\Delta}{ABD} \sim \overset{\Delta}{ACE}$$
Then;
$$\frac{|AB|}{|AC|} = \frac{|AD|}{|AE|}$$
$$\frac{4}{6} = \frac{|AD|}{|AD|+5} \Longrightarrow 4|AD| + 20 = 6|AD|$$
$$|AD| = 10\ cm$$
</td>
<td>


$$\overrightarrow{AB} = 2.\overrightarrow{BC}$$
$$\overrightarrow{DE} = 5\vec{v}\ and\ \overrightarrow{AD} = k.\vec{v}$$
$$\overrightarrow{AC} = k\vec{v} + 5\vec{v} + 9\vec{u}$$
$$\overrightarrow{AB} = k\vec{v} + 6\vec{u}$$
$$\overrightarrow{AC} = \frac{3}{2}\overrightarrow{AB}$$
$$k\vec{v} + 5\vec{v} + 9\vec{u} = \frac{3}{2}(k\vec{v} + 6\vec{u})$$
$$2k\vec{v} + 10\vec{v} + 18\vec{u} = 3k\vec{v} + 18\vec{u}$$
$$10\vec{v} = k\vec{v}$$
$$k = 10 \Longrightarrow \left\|\overrightarrow{AD}\right\| = 10\ cm$$
</td>
</tr>
</table>

A student who gains problem solving skills in geometry is expected to be able to associate the properties of geometric figures, make proofs and use coordinate plane and vectors in problem solving (Swings & Peterson, 1988). It is thought that the problem solving skills gained in geometry also support students to establish a relationship between geometry and the subfields of mathematics (numbers, measurement, algebra, probability and statistics) (MoNE, 2010).

Although SA, AA and VA, which are based on the connection of geometry with other fields in students' problem solving, have a place in geometry as a field of study, Euclidean geometry comes to mind almost all over the world when school geometry is mentioned (Dindyal, 2003). Geometry lessons are based on the SA consisting of the theorems developed by Euclid and their proofs (Kwon, 2012). For these reasons, students cannot be involved in an appropriate learning environment where they can make connections between their thoughts while solving problems or trying to prove a theorem. On the other hand, they encounter very few opportunities to unleash their creative abilities involving broader mathematical methods such as algebraic-analytical methods.

Considering the contributions that can be provided to students by using different approaches together in solving geometry problems, it is necessary to support geometry courses by using different approaches instead of focusing on only one approach. When the mathematics teacher education curriculum in Türkiye is examined, the courses that form the basis of the approaches (analytical geometry, geometry and linear algebra) are given in different years at the university level. It is seen that pre-service teachers who go through such an education process cannot be included in a course in which they will have the opportunity to establish a connection between the information belonging to the approaches. This situation reveals the idea that pre-service teachers are prevented from developing the skills of transferring knowledge and making connections between approaches. In general, the contents of these courses have a structure that is carried out only through the relevant approach. In particular, when the geometry course is examined, it is seen that a course content based on SA is carried out instead of using approaches that can offer different perspectives to students together. This situation shows us that the geometry course is in a structure that is far from providing the connectivity feature. Therefore, the necessity of a learning environment in which all three approaches will be used emerges. In this context, within the scope of this study, the content of the geometry course in the secondary mathematics teacher education programme was enriched with analytical and vector approaches by focusing on synthetic geometry and a learning environment was designed in which all three approaches were used together. The aim of this study is to examine the effect of the learning environment designed based on different approaches on the change in pre-service teachers' approach preferences and geometry achievement in geometry problems.

METHOD

Research Design

In this study, a case study was used since it was aimed to reveal the change in pre-service teachers' approach preferences in geometry problem solving after the designed learning environment. This study did not only focus on the pre-service teachers' preferences in the problem solving process, but also aimed to reveal the reasons underlying their preferences. Thus, it was also possible to reveal how and why the pre-service teachers' approach preferences changed when they were provided with a learning environment with more than one approach. On the other hand, as expected from pre-service teachers, this study provided an opportunity to examine the change in their problem solving achievements in an environment that would associate other areas of mathematics in geometry problems.

Context of the Learning Environment

While designing the learning environment based on AA, SA and VA. SA was taken as the basis and the process was supported with AA and VA. In the learning environment, the content was enriched by adding theorem proofs using SA and their implementations and geometry problems, and solutions to each theorem and problem with AA and VA. In the content applied for 10 weeks, 2 hours theoretical lesson was conducted with SA. The two-hour implementation process was designed in a way to enable the use of AA and VA in geometry problem solutions. Pre-service teachers were encouraged and guided to use other approaches along with SA while solving geometry problems. After each problem solution, the solutions made with all three approaches were written on the board and a discussion environment was created on the solutions. The model showing the designed learning environment is presented in Figure 1.

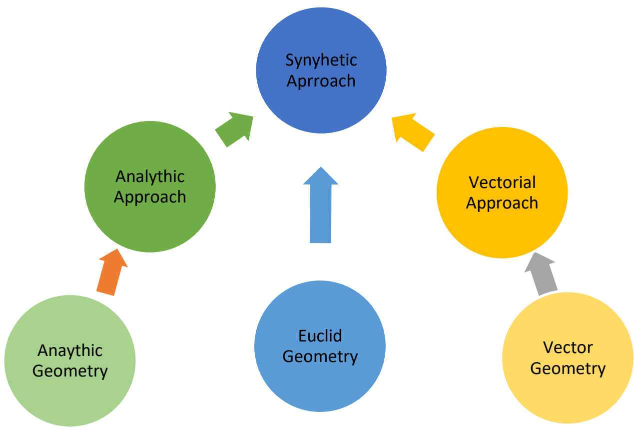


Figure 1. Learning Environment Model Based on AA, SA and VA

Research Sample

The sample of the study consisted of 20 prospective teachers studying in the secondary mathematics teaching program of a state university in Türkiye. Since modern learning theories argue that learning is based on prior knowledge, it is necessary for pre-service teachers to have prior knowledge of these approaches in order to use AA, SA and VA in geometry problem solving. In the learning environment designed to enable the use of AA, SA and VA together in geometry problem solving, pre-service teachers needed to have prior knowledge such as the distance between two points, the equation of a line given a point and its slope, the equation of a line given two points, the parallelism of lines with equal slopes, the product of the slopes of perpendicular lines, etc. in order to solve with AA. For this reason, the fact that the pre-service teachers who constituted the sample of the study had taken the Analytic Geometry course including this information in the undergraduate programme they studied was considered sufficient to provide the pre-competencies they should have for AA. In order to be able to use VA in solving geometry problems, pre-service teachers should have mastered the contents of topics such as vectors, matrices, determinants, systems of linear equations. This situation necessitated that pre-service teachers should have taken Linear Algebra-I and Linear Algebra-II courses including the above topics in order to have these pre-competencies. Only in this way, it could be accepted that pre-service teachers had the competence to use VA in problem solving. Therefore, it was important for the pre-service teachers forming the sample of the study to have taken Analytic Geometry and Linear Algebra courses, which form the basis of AA and VA, in order to be able to solve geometry problems with all three approaches in the designed learning environment. In this direction, while selecting the pre-service teachers constituting the sample of the study, it was paid attention that they had taken these courses. After the selection of pre-service teachers; analytical, synthetic and vectorial approaches were introduced to the group involved in the learning environment designed based on AA, SA and VA, the superior aspects of the approaches were discussed and problem examples using these approaches were presented.

Data Collection Tools

Two separate tests were created to reveal the geometry achievement of pre-service teachers and the approaches they prefer in geometry problem solving. While preparing the pre-test and post-test, various sources were utilized (Kisacanin, 2002; French, 2004; Aydın et al., 2011; MoNE, 2011; Ok, 2013). The problems used in these tests were evaluated by two researchers who are experts in the field of mathematics education. The experts have a doctorate degree in mathematics education. Evaluation meetings were held with the experts within the scope of the expert review method. In these meetings, the purpose of the study and the purpose of the achievement test to be developed were verbally expressed to the experts by the researcher. The problems and solutions selected for the tests were presented to the experts in writing and they were asked to evaluate the problems in terms of structure and content. After making the necessary examinations, the experts asked questions to the researcher, if any, and gave feedback on the suitability of this data collection tool. Thus, the internal validity of this data collection tool was ensured with expert opinions. These tests consist of six parallel problems. The pre-test was administered before the designed learning environment and the post-test was administered afterwards. Each of the geometry problems in these tests has solutions with all three approaches. For the problems in the tests, pre-service teachers were asked to make solutions with any approach they would choose. The pre-service teachers were given a total of 100 minutes to answer the tests.

With the clinical interviews conducted, it is aimed to determine the reasons for pre-service teachers to prefer the approaches they use in the problem solving process. The pre-service teachers selected for the clinical interviews were chosen from different achievement levels (high, medium, low) according to their test scores. During this selection, pre-service teachers' volunteering to participate in the interviews was also taken into consideration. The interviews conducted to determine the approach preferences of the pre-service teachers in their solutions were based on the problem solutions they made in the pre-test and post-test. During the interviews, pre-service teachers were asked questions such as "What is the reason for using the approach you preferred in problem solving?", "How did you determine that the approach you chose in problem solving was the most appropriate approach for this problem?", "Why did you need to change the approach you used in problem solving?". In the tests, the pre-service teachers were asked to explain how they reached the solutions they made with the approach they preferred for each problem. The interviews with the pre-service teachers were conducted individually and lasted approximately 45 minutes.

Data Analysis

The solutions given by the pre-service teachers to the problems in the tests were analyzed using the rubric developed by Malone et al. (1980). Some arrangements were made on the rubric by taking into account the aims of the study. This scoring key was preferred because the criteria in the scoring key were close to the steps that the prospective teachers went through in the problem solving process in this study. The original and revised versions of the rubric used for the analysis of the problem solving process are given in Table 1.

Table 1: Original Rubric and Revised Rubric Used for the Analysis

Original version of the rubric		Revised version of the rubric
Degree	Description	Description
0	The prospective teacher could not answer the question at all, wrote only what was given or wrote unnecessary expressions that were not used for the solution.	0-Point: The participant did not write anything about the solution, only what was given in the problem or expressions that did not contribute to the solution.
1	The prospective teacher wrote at least one necessary and valid statement for the solution and gave reasons for it.	1-Point: The participant wrote at least one necessary and valid statement for the solution of the problem and gave reasons for it.
2	The prospective teacher did almost half of the solution by using the appropriate chain of reasoning, but the solution was not completed due to the incorrect expressions used in the previous steps.	2-Point: The participant completed some steps in the problem solution correctly by using the appropriate reasoning chain, but could not complete the solution due to the incorrect expressions used in the previous steps.
3	The prospective teacher did all the steps of the solution correctly, but he/she made mistakes in the notations, words or names of the theorems he/she used during the solution.	3-Point: The participant completed the problem solution steps almost correctly, but made mistakes in the notations, words or names of theorems used during the solution.
4	The prospective teacher completed the solution with a maximum of one error in the representations.	4-Points: The participant has completed all the required solution steps.

In this scoring, the lowest score that prospective teachers can get as a result of solving a problem with each approach is 0, while the highest score is 4.

The problem solutions in the pretest and posttest were scored by two separate researchers according to the rubric. Afterwards, two researchers came together to re-evaluate the solutions with different scores and completed the scoring by reaching a consensus.

The data set obtained from the clinical interviews was coded by two separate researchers. In order to reach a consensus on the differences in the codes, the researchers came together to re-evaluate the data and the coding was finalized. In addition, in order to ensure the validity of the data, quotations from the clinical interviews conducted on the problem solutions made by the pre-service teachers were given in the paper.

FINDINGS

In this section, the approaches that pre-service teachers preferred while solving geometry problems before and after the designed learning environment and the reasons for their preferences were determined. Table 2 presents the approach preferences of pre-service teachers in solving the problems.

Table 2: Pre-service Teachers' Approach Preferences in Problem Solving

Approach	AA		SA		VA		No Answer		Total	
	n	%	n	%	n	%	n	%	n	%
Approach Preferences in the Pre-Test	8	6.7	79	65.8	10	8.3	23	19.2	120	100
Approach Preferences in the Post-Test	36	30	56	467	26	21.7	2	1.7	120	100

According to Table 2, when the solutions in the pre-test were analyzed, SA was preferred in 79 (65.8%) solutions, VA in 10 (8.3%) solutions and AA in 8 (6.7%) solutions. The remaining 23 (19.2%) solutions were left blank. According to the findings obtained from the pre-service teachers' approach preferences in problem solving before the implementation, it was determined that the most preferred approach was SA. The preference rate of this approach is quite high. The second most preferred approach was VA. The last preferred approach was AA.

According to the findings obtained after the implementation, it is seen that pre-service teachers preferred SA in 56 (46.7%) solutions, AA in 36 (30%) solutions and VA in 26 (21.7%) solutions. The remaining 2 (1.7%) solutions were left blank. After the implementation, it was determined that the most preferred approach for problem solving was SA. After the implementation, the second most preferred approach was AA and VA was preferred in the last place in problem solving. After the implementation, it was observed that the approach preferences were not concentrated in SA but also distributed to other approaches. It was also found that the number of solutions left blank decreased.

In order to examine the pre-service teachers' approach preferences before and after the implementation in more detail, clinical interviews were conducted with two pre-service teachers selected from each of the "good", "medium" and "low" levels determined in line with the scores obtained from the tests applied to the pre-service teachers. These pre-service teachers were coded as T-1, T-2, T-3, T-4, T-5 and T-6. T-1 and T-2 were selected from "low" achievement level, T-3 and T-4 from "medium" achievement level, and T-5 and T-6 from "high" achievement level. The interviews were conducted based on the problem solutions of the pre-service teachers in the tests.

Firstly, the reasons underlying the preference of pre-service teachers for AA before and after the implementation were analyzed. In this direction, according to the findings obtained from the interviews, the reason for pre-service teachers' preference for AA in problem solving is categorized under one code before the implementation and two codes after the implementation. Table 3 showing the frequency of these codes in the pre-service teachers who participated in the interviews is presented below.

Table 3: Frequency of Pre-service Teachers' Prevalence of the Codes Created for Preferring AA

	Code	T-1	T-2	T-3	T-4	T-5	T-6
Before the Implementation	Given in the Problem	1	1	-	2	1	-
After the Implementation	Placing the Geometric Shape on the Coordinate Plane	1	3	1	4	1	1
	Given in the Problem	-	2	-	1	2	-

Below are examples of pre-service teachers' views under these codes. The solution of a problem made by T-1 under the code "Given in the Problem" before the implementation and the reason for preferring AA in this solution are presented below.

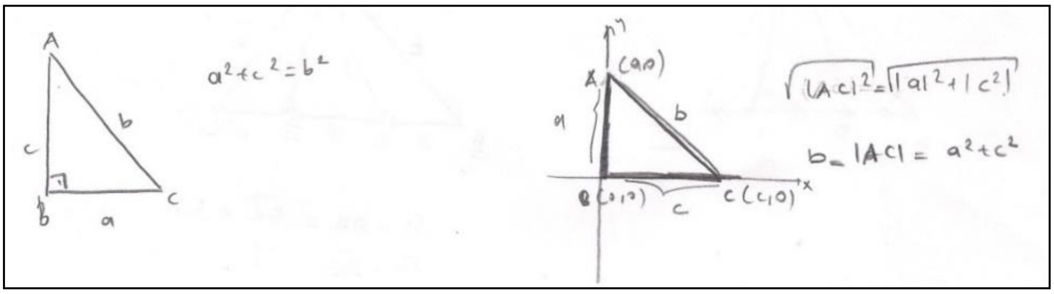


Figure 2. The Solution in which T-1 prefers AA

Researcher: You preferred to use the analytical approach in solving this problem. What is the reason?
T-1: I wrote the Pythagorean theorem here (showing the figure on the left). The expression in the Pythagorean theorem reminded me of the distance formula between two points in analytic geometry. For this reason, I placed any right triangle on the analytical plane and solved it.

As a result of the interview, the reason why T-1 preferred the analytical approach in problem solving was that the expression of the Pythagorean theorem given in the problem evoked the distance between two points in analytical geometry.

Another pre-service teacher who had an opinion under this code was T-4. The solution of the pre-service teacher and the dialogue carried out are presented below.

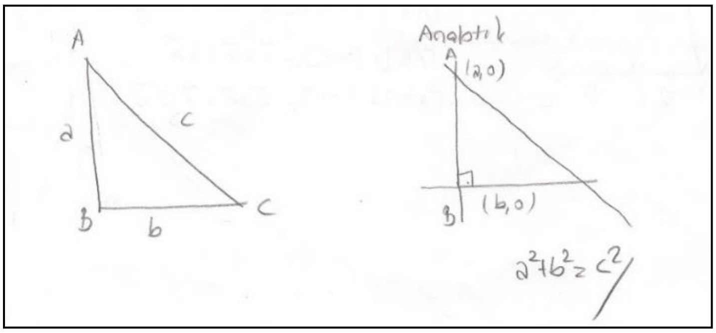


Figure 3. The Solution in which T-4 prefers AA

Researcher: You tried to do the Pythagorean theorem with the help of analytical approach. What is the reason that led you to this preference?
T-4: In the Pythagorean theorem, there is a right angle in the triangle. I thought it would be easier to place this shape on the coordinate plane.
Researcher: How did you think the right angle given in the question would be easier to use in the analytical plane?
T-4: If a right angle is given in a question, I think it is easier to use the analytical approach. Because when we place the shape on the coordinate plane using the starting point, the solution is seen more easily.

In the dialogue conducted over the analytical solution of T-4, the pre-service teacher has the opinion that it would be easier to solve the problem by placing the right-angled triangle in the Pythagorean theorem given in the problem on the analytical plane. It is thought that the solution will be much easier by placing the corner corresponding to the right angle of the right-angled triangle at the origin of the analytic plane. For these reasons, T-4 prefers to solve this problem with the analytical approach.

According to the findings obtained from the interview conducted after the implementation, the first reason for pre-service teachers to prefer the approaches they used in problem solving was that the geometric shape expressed in the problem could be easily placed in the coordinate plane. All pre-service teachers expressed their opinions under this code. The opinion expressed through the problem solution of T-3 is presented below. The solution of the pre-service teacher belongs to problem 1.

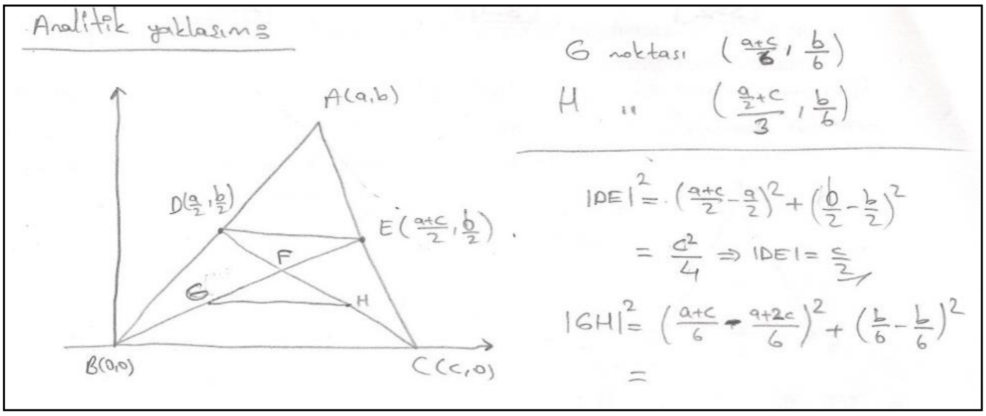


Figure 4. The Solution in which T-3 prefers AA

Researcher: What was the reason for preferring analytical approach in this problem?
T-3: There is an ABC triangle and I thought that if I place a corner of this triangle at the origin on the coordinate plane, the base of the triangle will be on the x-axis. In this way, it will be easy to place the corners of the triangle and the points given in the problem on the coordinate plane.
Researcher: Is there any other factor that pushed you to analytical approach in this problem?
T-3: As I said, one of the coordinates of the base side of the triangle will be (0,0), the other will be (c,0). I thought that the ratio of the sides required in the problem could be found very easily from the distance between the two points. Therefore, I used this approach.

The geometric shape given in the problem had a great effect on the T-3's preference for AA in his solution. The pre-service teacher stated that it was quite easy to place a triangle on the coordinate plane and to determine the vertices of this geometric shape on the coordinate plane. The opinions that determining the base on the x-axis after placing a corner of the triangle to the origin on the coordinate plane contributed to the operations in the problem solution also pushed the pre-service teacher to prefer AA. On the other hand, the fact that the lengths of the two line segments to be found in the problem were equal reminded the distance formula between two points in the coordinate plane. For these reasons, the pre-service teacher preferred to use AA in the solution.

In the code of "Given in the problem" determined after the implementation, the opinions of T-2, T-4 and T-5 are available. While this code was formed, it was determined that the expressions given in the problems in the test were effective in preference of the approach. The pre-service teachers' preference for the approach as a result of the association of some concepts or expressions with the concepts in the fields that form the basis of the approaches (analytical geometry, Euclidean geometry, vector algebra) were collected under this code. In order to give an example of this situation, the statement belonging to T-2 is given below.

> *T-2: If there is perpendicularity in the question, I use analytical approach because the perpendicularity in the analytical plane makes me feel that this approach can be more useful for the solution. Thus, I think that the solution will be a little easier. I used analytical approach in such questions.*

According to the above statement of T-2, a connection was established between the properties of the coordinate system, which is the basic element used in the field of analytical geometry, and the properties given in the problem. It is stated by T-2 that an easier solution can be followed by placing the corner where the right angle in the right triangle in the related problem at the starting point in the coordinate system. Thus, one of the reasons for preferring the approach used in problem solving is that some concepts or expressions given in the problem evoke the expressions and concepts used in the related approach. For this reason, the expressions given in the problem play an active role in pre-service teachers' preference for the approach they will use in their solutions.

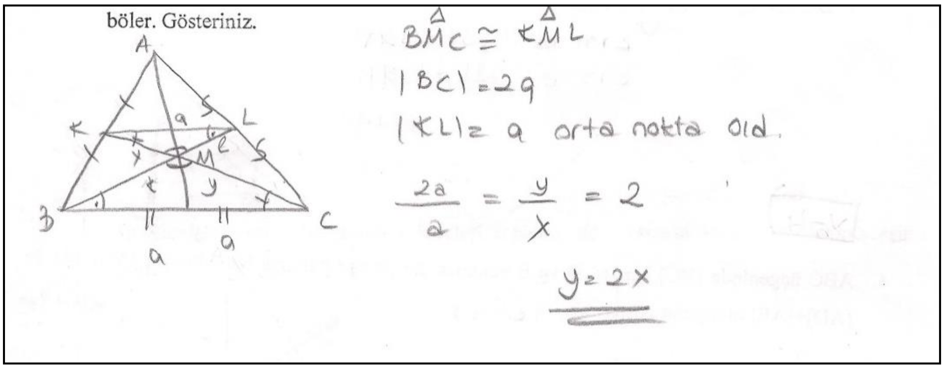
The reasons for the preference of SA, which is the most preferred approach by pre-service teachers before and after the implementation, are gathered under 5 codes before the implementation and under 3 codes after the implementation. Table 4 showing the frequency of the codes determined for the reasons for pre-service teachers' preference for SA before and after the implementation is presented below.

Table 4: Frequency of Pre-service Teachers' Prevalence of the Codes Created for Preferring SA

	Codes	T-1	T-2	T-3	T-4	T-5	T-6
Before the Implementation	Having Experience	4	2	1	4	2	4
	Newness of Other Approaches	1	2	-	1	2	5
	Given in the Problem	-	3	-	4	2	2
	Preventing Mathematical Operation Intensity	1	2	-	1	-	-
	SA is Necessary to Solve Problems with Other Approaches	-	1	-	-	-	1
After the Implementation	Habituation to SA	3	-	1	2	2	3
	Given in the Problem	-	-	3	1	1	1
	Feeling Safe	-	1	-	2	-	2

When Table 4 is analyzed, the code containing the highest number of pre-service teachers' opinions under the codes consisting of the reasons for pre-service teachers to prefer SA in problem solving before the implementation is "Having Experience". This code includes the opinions of all pre-service teachers. There are five pre-service teachers' opinions under the second code. In the third code in the table, there are four pre-service teachers' opinions. The frequency of pre-service teachers' opinions under the second and third codes is the same. Three pre-service teachers expressed their opinions for the fourth code, "Preventing Mathematical Operation Intensity". For the last code, only two pre-service teachers expressed one opinion. This code has the lowest frequency of teacher opinions. Below, each of these codes is examined under subheadings and the dialogues and sample question solutions are given.

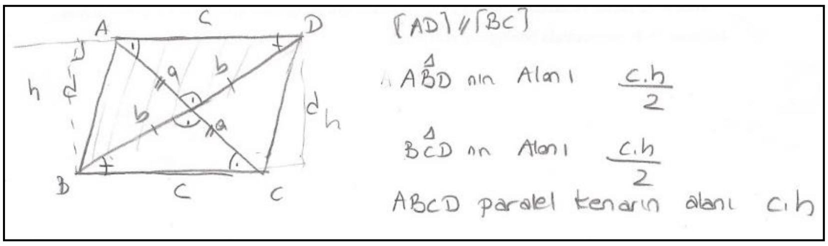
Among the reasons why pre-service teachers prefer SA in problem solving processes, the most frequently stated reason is that they have experience in SA. All of the pre-service teachers stated that they had seen the SA before and that they were closer to this approach because they had solved geometry problems only with this approach so far. T-4 made the following statements about his preference for the SA in problem solving.



Researcher: What are the reasons that led you to solve the problem with SA in the above problem?
T-4: It was easy for me to draw parallel on the figure. Then I used interior opposite angles from parallelism. From there, I used similarity. Since these were things we already knew before, it was easier.
Researcher: Did you use the SA because of parallelism?
T-4: We have always used parallelism on the figure with the SA before. For this reason, I preferred this approach. We had emphasized on these issues in geometry when we were in high school. For this reason, it is more practical and easier to make solutions with this approach.

T-4 stated that the reason for preferring the SA was that the parallelism feature in the shape of the related problem was always solved with the SA based on his previous experiences. When the pre-service teacher was asked whether only the concept of parallelism in the shape led him to the SA, the pre-service teacher emphasized that he had experience in previous years that these and similar problems were solved with the SA.

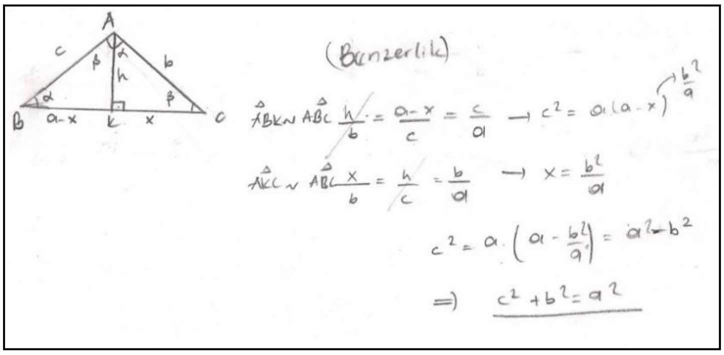
"Given in the Problem" code ranked third among the reasons for pre-service teachers to prefer SA. Under this code, the solution of T-4 and the reason for preference are presented below.



Researcher: You used the SA in this problem. What did you want to do in the solution?
T-4: I tried to go from the area, so I felt that I had to reduce the perpendicular to one side of the rhombus. But I couldn't do it. The situation of reducing the perpendicular here reminded me of the properties we use in SA. For this reason, I preferred to use SA in this solution.

In the solution, the pre-service teacher perceived the word "square" in the expression "squares of the sides" as finding the area of the shape and stated that the area of a shape can be found with a synthetic approach.

One of the underlying reasons for the pre-service teachers to use the SA while solving geometry problems was determined as "Preventing Mathematical Operation Intensity". There are two pre-service teachers' opinions under this code. The first of these belongs to T-2. The problem solution of the pre-service teacher and the dialogue on this solution are presented below.



Researcher: What prompted you to use the SA in this problem?
T-2: I think that the synthetic approach allows me to make the similarity in an unprocessed way.
Researcher: What do you mean by operation?
T-2: For example, in the VA, we can ensure that two edges are perpendicular to each other when the inner product of the vectors representing the edges is zero. In the AA, we need to place each point in the figure on the analytical plane and then perform the operation. I think these are computational overheads. In the SA, we can directly write the ratios in similarity. For this reason, I solved with the SA.

T-2 states that the SA gives the result directly. The pre-service teacher, who argues that it is necessary to perform many extra operations to reach the result in AA and VA, states that he prefers the synthetic approach used in the solution of this problem because it prevents redundancy of operations.

After the implementation, the code with the highest number of pre-service teachers' opinions was "Habituation to SA". Only T-2 coded pre-service teacher did not express any opinion for this code. The code "Given in the Problem", which consists of the opinions of four pre-service teachers in total, ranked second. In the last place is the "Feeling Safe" code consisting of the opinions of three pre-service teachers. Below, examples of teacher opinions in these codes will be presented with problem solutions respectively.

The opinions of T-1 under the code of "Habituation to SA" are presented below.

T-1: If I had been taught to solve geometry problems with the VA since my childhood, I would have done it very well. This time the SA would have been foreign to me. If I had learnt the analytical approach at first, the others would have been foreign to me. I think it is nothing but habit. If we had learnt the VA first and not the others, if we had been told that geometry consists of vectors, we would have solved problems very well with the VA. That would have been the case. But for years we have always used the same similarities etc. We always see the same things and this has become like reading and writing. The information in the SA is settling down. The information we use in other approaches is as if it is new information.

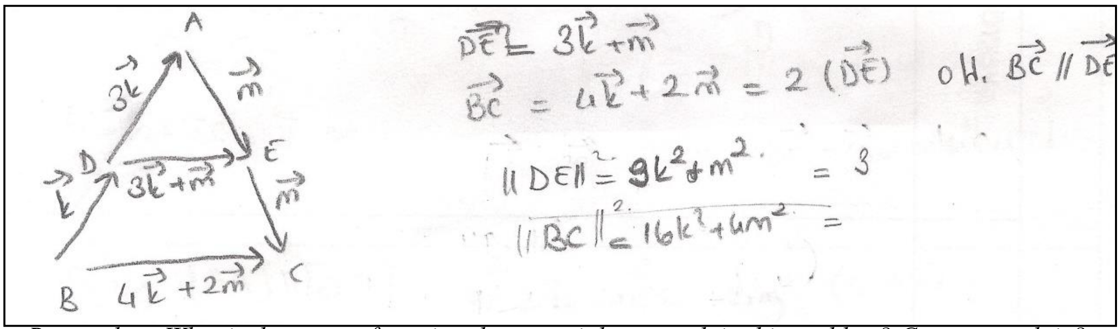
Finally, the reasons underlying the pre-service teachers' preference for VA before and after the implementation were analyzed. In this direction, the reasons for pre-service teachers' preference for VA in problem solving before and after the implementation are collected in one code. The frequency of these codes in the pre-service teachers who participated in the interviews is presented in Table 5.

Table 5: Frequency of Pre-service Teachers' Prevalence of the Codes Created for Preferring AA

	Code	T-1	T-2	T-3	T-4	T-5	T-6
Before the Implementation	Using the Properties of Vectors	1	-	-	1	1	-
After the Implementation	Using the Properties of Vectors	1	1	1	3	2	2

When Table 5 is analyzed, the reasons why pre-service teachers preferred VA before and after the implementation are gathered under a single code. Under this code, there are three opinions of pre-service teachers before the implementation and six opinions of pre-service teachers after the implementation. The dialogues conducted with the pre-service teachers and sample problem solutions are given below as examples of the opinions that constitute this code.

Before the implementation, the solution of T-1 for the first problem in the test and the reason for preferring the vectorial approach in this solution are presented below as a dialogue.

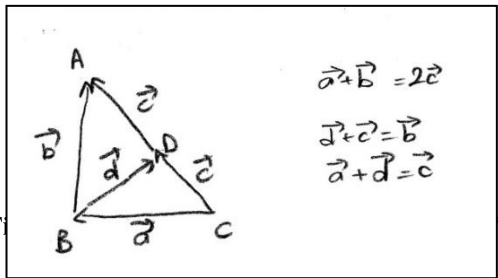


Researcher: What is the reason for using the vectorial approach in this problem? Can you explain?
T-1: Firstly, I drew the given shape with the help of vectors. I tried to write the sides as the sum of the vectors. I think it was easy for me to solve the problem by using the sum of the vectors.

According to the dialogue between the researcher and the pre-service teacher above, T-1 shows that the edges are parallel to each other through the sums of vectors by using the end-to-end addition method from the properties of vectors in the solution she made for the related problem. The pre-service teacher stated that it was easier to solve this problem by using the properties of vectors and preferred the vectorial approach because she could use the properties of vectors.

After the implementation, it was observed that all pre-service teachers had at least one opinion that would form the code "Using the properties of vectors". In these opinions, pre-service teachers want to prefer VA when they think that they can use the properties of vectors that are frequently used in vectorial approach. These properties of vectors include; addition with vectors, determining the directions of vectors, length of vectors, finding the areas of geometric shapes by means of vectors, etc. Pre-service teachers prefer this approach when they think that they can use the vector properties given above in solutions.

The opinions of the pre-service teacher coded T-5 about this code are presented below as an example.



Fi [illegible] A

T-5: I tried to draw the right triangle given in the problem with the help of vectors. I used the end-to-end addition method. Thus, I was able to easily create both a large triangle and two small triangles. Based on the shape I created, I wrote the vectors as the sum of other vectors. After all, I know these properties of vectors and it is very easy to use them. For this reason, I wanted to solve with VA.

In the solution of the related problem, T-5 drew the triangle in Figure 9 by using the end-to-end addition method in vectors. He expressed the resulting right triangle ABC and the sides of the triangles BDA, CBD as the sum of the vectors. In line with the opinions received from T-5, he preferred to use this approach because he already knew the vector properties he used in her operations. It can be said that the T-5's knowledge of vectors positively affected their preference for VA in problem solving.

In this study, which examines the approach preferences of pre-service teachers before and after the designed learning environment, it is important for the significance of the study to give their achievement status in the tests. In the other words, not only the change in pre-service teachers' preferences but also the extent to which this situation is reflected in their achievement was determined in this way. Table 6, which shows the achievement status of the pre-service teachers in the pre and post-tests, is given below.

Table 6: Pre-test and Post-test Scores of Pre-service Teachers

		Pre-Test		Post-Test		Total	
		n	%	n	%	n	%
Scores	0-Point	79	65.9	5	4	84	35
	1-Point	17	14.2	1	0.8	18	7.5
	2-Point	4	3.3	12	10.1	16	6.6
	3-Point	10	8.3	27	22.4	37	15.4
	4-Point	10	8.3	75	62.7	85	35.5

When Table 6 is examined, it is seen that in 10 (8.3%) of the solutions in the pre-test, the pre-service teachers were successful by completing all steps correctly. In line with this finding, it is seen that the geometry achievement of the pre-service teachers was quite low before the implementation. In the remaining 110 (91.7%) solutions, the pre-service teachers were considered unsuccessful because they could not complete the problem solving steps correctly. In 75 (62.5%) of the solutions in the post-test, it is seen that the pre-service teachers were successful by completing all steps correctly in their solutions. It was determined that there was an increase in the geometry achievement of the pre-service teachers in the post-test compared to the pre-test. In the remaining 45 (37.5%) solutions, the pre-service teachers were unsuccessful. As a result, it can be said that different approach situations in problem solving presented to pre-service teachers positively affected their geometry problem solving achievement.

CONCLUSIONS AND DISCUSSION

In this study, which focused on the change in pre-service teachers' geometry problem solving preferences due to the learning environment designed based on different approaches, the change in their geometry achievement was also examined.

Before the designed learning environment, the percentages of pre-service teachers preferring AA, SA and VA in problem solving were 6.7%, 65.8% and 8.3%, respectively. The remaining 19.2% was left unanswered. It can be concluded that students who have received an education based on Euclidean geometry throughout their entire education life naturally preferred SA. Similar opinions of pre-service teachers were also obtained in the study. The pre-service teachers stated that they felt comfortable solving with SA, which they were mostly used to. Although they preferred solving with AA and VA, this rate was found to be low. In fact, during the selection of the pre-service teachers, attention was paid to the fact that they had taken Analytic Geometry and Linear Algebra courses. This shows that although the pre-service teachers had sufficient knowledge for AA and VA, they were limited in their preference for these approaches. In some problems, pre-service teachers who thought that the solution process with AA and VA was a difficult and time-consuming process and that it was sufficient to reach a solution with a single approach preferred SA. Similarly, Aydın-Güç (2015) states that in the problem solving process, pre-service teachers think that a single solution method is sufficient and that they do not need to deal with different solution methods. Dowlath (2008), on the other hand, stated that although pre-service teachers know that there are different strategies for solving real life problems, they prefer to use the most familiar strategy for the solution.

After the implementation, the percentages of pre-service teachers' preference for AA, SA and VA were 30%, 46.7% and 21.7%, respectively. 1.7% of the solutions were left blank. The change in the approach preferences of the pre-service teachers who participated in the designed learning environment in problem solving processes is remarkable. SA, which was the most preferred approach before the implementation, was also preferred in the first place after the implementation. From this point, it is seen that the pre-service teachers' tendency to use SA in synthetic geometry problems continues. Although the preference rates of other approaches have increased, almost half of the pre-service teachers still prefer SA in problem solving. This may be thought to be due to the fact that they have difficulties in the process of adapting to a new situation for them. Because accepting and adapting to a new situation is accepted as a process that requires a long time. For this reason, it is thought that although the practices in the environment were continued for a period of time, this period could not have a sufficient effect on the problem solving behaviors of the individuals. It was also found that efforts to encourage the use of other approaches had short-term effects. When another problem was encountered, most of the solutions were made by preferring SA. A similar situation is observed in the study of Gagatsis and Demetriadou (2001). In this study, it was concluded that classical geometry (synthetic geometry), which is generally used in high school geometry courses in other countries, had a strong influence on Greek students' preferences in geometry problem solving. Allendoerfer (1969) also emphasizes the tendency towards traditional geometry (synthetic geometry) in his study. Nissen (2000) examined the situations in which different approaches were used in geometry problem solving and found that SA was the most widely known and used approach by students. On the other hand, in the interviews conducted, it is another remarkable situation that some students see SA as the starting point for solving with AA and VA. One pre-service teacher stated that the prerequisite for solving the problem with AA or VA was to have solved the problem with SA. It was observed that some pre-service teachers who stated that they could use other approaches based on the synthetic solution were able to transfer their knowledge in SA to AA and VA. However, on the contrary, it is a fact that there are problems in transferring the knowledge in AA or VA to SA in the problem solving process. In this context, it reveals the idea that the designed environment is insufficient at the point of transferring knowledge between approaches.

The small difference between the preference rates of AA and VA before the implementation increased in favor of AA after the implementation. One of the reasons underlying the increase in AA, which was the least preferred approach before the implementation, is that it is generally sufficient to use simple knowledge in the analytical solutions of the problems used. It was also determined that there were opinions supporting this situation in the interviews. In particular, the number of pre-service teachers who prefer this approach because it gives short results is quite high. Considering the preference rate of AA, it is thought that the designed learning environment contributed to the behavior of using it at the desired level in the problem solving process. After the implementation, it is seen that the least preferred approach is VA. Although the vector solutions of the problems generally require a basic level of knowledge, it is seen that the development of pre-service teachers in VA is resistant. It is thought that this resistance is generally due to the experiences of pre-service teachers in problem solving processes throughout their education. In addition, it can also be said that pre-service teachers prefer VA in the last place due to the limited number of problems solved in the designed learning environment or the fact that the solutions in VA usually involve higher level reasoning skills. On the other hand, it is thought that the experiences in the learning environment also have an effect on the preference for this approach. While the pre-service teachers used VA in the implementations in the designed environment, they had problems especially in the part of constructing a geometric shape by means of vectors. Due to the fact that this deficiency could not be avoided, pre-service teachers hesitated to use VA, especially in problems where geometric shapes should be used for the solution. This situation was effective in the perception of pre-service teachers that it is difficult to use VA in geometry problem solving. A similar situation was found in Kwon's (2012) study. In the study, it was stated that students were reluctant to solve problems with VA because they thought that solving problems with vectors was difficult. In some problem solutions, it was observed that pre-service teachers could not reach the result in solutions where the sum of vectors was used. This process can also be considered as a factor for pre-service teachers not to prefer VA. As a matter of fact, it was revealed in the interviews that the pre-service teachers had an aversion to this approach due to the mistakes they made in the operations with vectors. Gagatsis and Demetriadou (2001) identified similar errors in their study and stated that students preferred VA much less than SA in problem solving due to the errors they made.

The designed learning environment contributed positively to pre-service teachers' geometry achievement. It was determined that the success of the pre-service teachers in the post-test solutions increased considerably compared to the pre-test. It can be thought that using different approaches in geometry problems eliminates the limitation that students experience when using only SY and provides the flexibility to use other approaches when they cannot reach the result with any approach. In this way, students' geometry problem solving success increases. Similar findings have been found in previous studies. These studies (Schoenfeld, 1983; Levav-Waynberg, 2011; Pehlivan, 2011; Kwon, 2012) show that using different ways, strategies or approaches in problem solving increases problem solving skills and success.

Limitations and Future Research

This study aims to examine the effect of the learning environment designed based on AA, SA and VA on pre-service teachers' approach preferences and geometry achievement in solving geometry problems. Although significant findings were obtained, the study has some limitations.

The content of the learning environment designed based on the use of different approaches in geometry problem solving was limited to the axiomatic structure of geometry, triangles, quadrilaterals and circles in the undergraduate geometry course. The reason for this is that SA is the basis of the learning environment design and the alternative solutions of the problems in these topics in the content of the course are shown more easily with AA and VA.

The generalizability of the findings in this study is limited to the problem solutions of 21 pre-service teachers. The reason for this is that the prospective teachers should have some criteria (such as having taken Analytic geometry and Linear Algebra courses before). In addition, in order to be included in the process, the pre-service teachers had to respond to both the pre-test and the post-test. Under these conditions, a total of 21 pre-service teachers were selected as participants.

Another limitation is that the duration of the learning environment was limited to 10 weeks. The geometry course at the undergraduate level lasts 14 weeks in total and data collection processes are added to this process in the designed learning environment. Therefore, the duration of the designed learning environment was determined in this way.

The learning environment designed within the scope of the research was created by using AA, SA and VA together to enrich the geometry course content in the undergraduate mathematics teaching program. The learning environment designed in this study contributed to the pre-service teachers to utilize different approaches in solving Euclidean geometry problems and to solve the problems successfully. On the other hand, pre-service teachers who participated in this environment had a productive process in transferring knowledge between approaches. In this context, it is recommended that courses with similar content to the learning environment designed in this study should be included in mathematics teacher training programs.

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